

ESTIMATION OF O-D MATRICES: THE RELATIONSHIP BETWEEN PRACTICAL AND THEORETICAL CONSIDERATIONS

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ABSTRACT

The paper provides a comprehensive overview of the entire class of formulations and most recognized solutions for estimating Origin-Destination (O-D) demand tables. Specifically, the paper compares trip distribution formulations to less known synthetic O-D estimation solution techniques. The paper demonstrates that the trip distribution gravity model is nothing but a subset of the maximum likelihood solution to the synthetic O-D problem. Finally, the paper proposes a numerical solution to the maximum likelihood synthetic O-D problem that overcomes the shortcomings of the state-of-practice formulations. The proposed solution, which has been implemented in the QUEENSOD software, does not require flow continuity at the network nodes as is the case for most formulations. The latest version of QUEENSOD can be showed to yield results consistent with standard equation solvers (Excel and Matlab) and with exhaustive enumeration, for small networks, where all three methods are feasible. However, QUEENSOD has also been applied to networks with more than 1,000 zones and 5,000 links on a PC, where it typically requires no more than 1 hour to solve problems of this size. The solutions obtained by QUEENSOD reflect multi-path routings, consider correctly that the total number of trips in the network may not be constant, and properly reflect the role of the seed matrix. The model can also be applied to deal with problems where the routes are not known a priori.

INTRODUCTION

A common class of traffic engineering and transportation planning problems revolves around the estimation of an Origin-Destination (O-D) matrix for a network. This class of problems can take on many different formats, depending upon what is considered to be known and what assumptions are made to derive the missing parameters. This paper provides a comprehensive overview of this entire class of problems and emphasizes both the various problem formulations and numerical solution approaches that can be considered. Furthermore, the paper proposes a numerical solution to the synthetic O-D problem that overcomes the shortcomings of state-of-practice maximum likelihood formulations. The objective of the paper is to provide an overview to potential users of the available O-D estimation techniques in such a manner as to permit them to make intelligent decisions as to which technique to utilize and to appreciate the implications of the approximations that are commonly made.

Overview of O-D Estimation Problem

The first categorization, of the available estimation techniques, relates to whether the O-D's to be estimated are static, and apply to only one observation time period, or whether estimates are required for a series of linked dynamic time periods. This paper will focus primarily on the static or single time period problem. The next breakdown relates to whether the estimation is based on information about the magnitude of trip ends only, or whether information is available on additional links along the route of each trip. The former problem is commonly referred to as the trip distribution problem in demand forecasting, while the latter problem is commonly referred to as the synthetic O-D generation problem. This paper applies to both problems, but will focus on the latter synthetic O-D generation problem. The former is viewed simply a simpler subset of the former.

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Within the overall static synthetic O-D generation problem, there are two main flavors. The first exists when the routes that vehicles take through the network are known a priori. The second arises when these routes need to be estimated concurrently while the O-D is being estimated. A priori knowledge of routes can arise automatically when there is only one feasible route between each O-D pair, or when observed traffic volumes are only provided for the zone connectors at the origins and destinations in the network. The first condition is common when O-D's are estimated for a single intersection or arterial, or a single interchange or freeway. The second condition is the default for any trip distribution analysis. This paper will focus on situations where the routes are known a priori, but it should be noted that a solution to the more general problem does exist. The more general solution involves an iterative use of the solution approach when routes are known a priori.

Within the static synthetic O-D generation problem, for scenarios where routes are known a priori (or are assumed to be known a priori) there exist two sub-problems. The first of these problems relates to situations where flow continuity exists at each node in the network, and multiple O-D matrices can be shown to match these observed flows exactly. In this case, the most likely of these multiple O-D matrices needs to be identified. The second sub-problem relates to situations where flow continuity does not exist at either the node level or at the network level. In other words, the observed traffic flows are such that no matrix exists that will match the observed flows exactly. In this case, a new set of complementary link flows are identified that maintain flow continuity by introducing minimum alterations to observed flows. Once such a complementary set of flows has been identified, the maximum likelihood problem can be solved as before.

The static synthetic O-D generation problem, for scenarios where flow continuity does exist, can be formulated in two different ways. The first of these considers that the fundamental unit of measure is the individual trip, while the second considers that the fundamental unit of measure is the observation of a single vehicle on a particular link. The availability of a seed or target O-D matrix is implicit in the latter formulation, but can be dropped in the former formulation. However, only when a seed matrix is properly included in the former formulation is it guaranteed to yield consistent results with the latter formulation. In other words, the absence of a seed matrix in the trip based formulation can be shown to yield inconsistent results, at least for some networks in which the multiple solutions result in a different number of total trips.

An additional and related attribute, of the trip-based formulation of maximum likelihood, is the presence of a term in the objective function that is based on the total number of trips in the network. This term, referred to as T , is often dropped in some approximations. However, it can be shown that dropping this term can yield solutions that represent only a very poor approximation to the true solution. In contrast, approximations involve the use of Stirling's approximation, for representing the logarithm of factorials, are shown to yield consistently very good approximations. This finding is critical because use of Stirling's approximation is critical to being able to compute the derivatives that are needed to numerically solve the problem (it is difficult to take derivatives of terms that include factorials).

Once Stirling's common approximation has been made, further approximations have been made by Willumsen (1978) and Van Zuylen and Willumsen (1980). These eventually permit the synthetic O-D generation problem to be reduced to a linear regression problem. While these approximations provide for solutions that are easier to compute, these solutions, except for the Stirling's approximation, are also shown to considerably degrade the accuracy of the final solution, when compared to the formulation without the approximation.

The paper is concluded by advocating the solution to the maximum likelihood problem by finding the solution to a complex set of non-linear equations. Within this set of equations, both the number of unknowns and the number of equations are both equal to 2 times the square of the number of zones in the network for which the O-D is being estimated. For a typical network with say 1,000 zones, this implies 2,000,000 unknowns and 2,000,000 equations. The above solution can be demonstrated to yield very accurate results for small networks, where the actual solution can be verified independently through exhaustive enumeration of all possible solutions. Furthermore, for small networks, the above equations can be solved using most standard equation solvers, such as those in Excel or MATLAB. However, for network problems, whose scope is beyond these standard equation solvers, a very efficient special-purpose equation solver has been developed, called QUEENSOD (M. Van Aerde & Assoc., 2002).

The latest version of QUEENSOD (Version 2.10) can be showed to yield results consistent with these standard equation solvers and with exhaustive enumeration, for small networks, where all three methods are feasible. However, QUEENSOD has also been applied to networks with more than 1,000 zones and 5,000 links on a PC, where it typically requires no

more than 1 hour to solve problems of this size. The solutions obtained by QUEENSOD reflect multi-path routings, consider correctly that the total number of trips in the network may not be constant, and properly reflect the role of the seed matrix. The model can also be applied to deal with problems where the routes are not known a priori.

Paper Contribution

The contributions of this paper are three-fold. First, the paper demonstrates that gravity model trip distribution procedures can be considered a subset of more generalized synthetic O-D estimation procedures. Second, the paper presents the various assumptions that are involved in solving the static O-D problem using maximum likelihood, as proposed in the literature. Third, the paper presents a new procedure for solving the static O-D problem that only requires the Stirling's approximation. This proposed procedure, which is incorporated in the QUEENSOD software, is applied to small networks to demonstrate its capabilities and moreover has been successfully applied to large urbanized networks.

Paper Layout

The following sections of the paper will provide a more detailed discussion of the nature of each of the above facets of the O-D estimation problem, and will describe their implications for the accuracy or validity of the final O-D estimate. Initially, the relationship between the trip distribution gravity model and synthetic O-D estimation is described in order to demonstrate the consistency in solution approaches. Subsequently, the under-specification nature of the O-D problem is discussed with the associated multiple solutions. The need to select the most likely of these O-D solutions is also discussed using either the trip versus link flow approach. The issue of flow continuity is discussed and the various approximations that are made to solve the O-D problem are overviewed. Finally, the conclusions of the paper are presented.

TRIP DISTRIBUTION VERSUS SYNTHETIC O-D GENERATION

Several methods are used for trip distribution including the gravity, growth factor, and intervening opportunities models. The gravity model is most utilized because it uses the attributes of the transportation system and land-use characteristics and has been calibrated and applied extensively to the modeling of numerous urban areas. The model assumes that the number of trips between two zones is directly proportional to the number of trip attractions to the destination zone and inversely proportional to a function of travel time between the two zones, as demonstrated in Equation 1.

Typically the values of P_i and A_j are computed based on trip distribution procedures. The values of F_{ij} are computed using a calibration procedure that involves matching modeled and field trip length distributions. K_j values are used when the estimated trip interchange must be adjusted to ensure that it agrees with observed trips by attempting to account for factors other than travel time. The values of K are determined in the calibration process, but considered judiciously when a zone is considered to possess unique characteristics.

$$T_{ij} = P_i \left(\frac{A_j F_{ij} K_j}{\sum_j A_j F_{ij} K_j} \right) \quad [1]$$

A comparison of the gravity model and the synthetic O-D solution approaches is best described through an example illustration. Specifically, consider the 16-link network illustrated in Figure 1, which illustrates a set up in which there are 4 zones (A, B, C and D), and 16 links. Links 1, 3, 5 and 7 represent zone connectors leaving each zone, while links 2, 4, 6 and 8 represent zone connectors entering each zone. Furthermore, links 11, 13, 15 and 17 represent links that provide for clock-wise flow around the network, while links 12, 14, 16 and 18 provide for counter clockwise flow.

If the above problem were to be viewed as a trip distribution problem, traffic flows on links 1 to 8 would only be available. Specifically, links 1, 3, 5 and 7 would represent zonal trip productions, while links 2, 4, 6 and 8 would represent trip attractions. Within the normal trip distribution process, information on links 11 through 18 would not be considered explicitly. The absence, of any consideration of any en-route link flows also makes it unimportant to know the routing of

traffic from, say A to D. In contrast, knowledge of the routes is critical during the synthetic O-D generation process, as the relevance of link counts to a particular O-D pair is highly dependent on the fraction of vehicles from that O-D pair that cross that link.

The complementary nature of the synthetic O-D generation process and the trip distribution process is best demonstrated in Table 1. Note that it is assumed that the zone connectors are dummy links, rather than actual streets or groups of streets. Therefore they are not included in the synthetic O-D generation process as inputs. In contrast, if these zones were well defined subdivisions, business or industrial parks, or shopping centers, the zone connectors could represent actual or physical roads, and could therefore be counted, and included in the synthetic O-D generation process.

Similarly, it is considered that for the trip distribution process, the trip production and attraction rates are not measured using road counts, but are estimated from land use based on trip distribution equations. Such forecasts are typically estimated as daily averages, and sometimes as peak period averages. However, at present, trip end estimates are not commonly available by 15-minute interval, making the estimation of 15-minute O-D matrices using trip distribution methods is virtually impossible. Of course, planning scale 24-hour or peak period matrices can be scaled to produce 15-minute matrices using traffic flow counts. However, a better approach would be to treat the trip distribution matrix as a seed matrix, and then systematically utilize the observed traffic flow counts in a synthetic O-D generation process.

Because the O-D problem is under-specified, multiple O-D demands can generate identical link flows, as will be discussed in more detail in the forthcoming section. The use of a seed matrix ensures that the optimum solution that that satisfies the link flows resembles the seed solution as closely as possible. Specifically, the use of a seed matrix that is computed using Equation 2 results in a final solution that is very similar to the gravity model solution, as demonstrated in Table 2.

$$t_{ij} = T \frac{F_{ij}}{\sum_{ij} F_{ij}} \quad [2]$$

MULTIPLE SOLUTIONS

Both the synthetic O-D generation process and the trip distribution process are subject to an information under-specification limitation. For example, if one attempts to estimate an O-D matrix for a 100 zone network with, say 1000 links, one has more unknowns to solve for than there are constraints. In the case of the trip distribution process, there are 100x100 O-D cells to be estimate, and only 2x100 trip end constraints. In the case of the synthetic O-D generation process, there are again 100x100 O-D cells to estimate, and only 1000 link constraints. Given the possibility of multiple solutions, both the trip distribution process and the synthetic O-D generation process invoke additional considerations to select a preferred matrix from among the multiple solutions, as indicated next.

In the case of synthetic O-D generation, the desire is to select from among all of the possible solutions, the most likely. This approach requires one to define a measure of the likelihood of each matrix. In general, there are two approaches to establish the likelihood of a matrix. One of them treats the trip as the basic unit of observation, while the other considers a volume count as the basic unit of observation. Both approaches will be discussed in greater detail later, but for now it suffices to indicate that for any matrix with cells T_{ij} , the likelihood of the matrix can be estimated using a function $L=f(T_{ij}, t_{ij})$, where t_{ij} represents prior information. The prior information is often referred to as a seed matrix, and can be derived from a previous study or survey. In the absence of such prior information, all of the cells in this prior matrix should be set to a uniform set of values.

In the case of the trip distribution process, the additional information that is added is some form of impedance. For example, the original gravity model considered that the likelihood of trips between two zones was proportional to the inverse of the square of the distance between the two zones. Since that time, many more sophisticated forms of impedance have been considered, but for the purposes of this discussion, all of these variations can be generalized as being of the form F_{ij} , where $F_{ij} = f(c_{ij})$ or the generalized cost of inter-zonal travel. What is less obvious, however, is the fact that the use of this set of impedance factors F_{ij} , is essentially equivalent to the use of a seed matrix t_{ij} .

This implies that solving the trip distribution problem, using zonal trip productions and attractions as constraints, together with a trip impedance matrix, is essentially the same as solving the synthetic O-D generation problem using zone connector in and out flows as constraints, and utilizing a seed matrix based on Equation 2, which is sometimes utilized in the gravity model trip distribution procedure. This similarity is very encouraging to most traffic engineers and planners for the following reasons. First, many planners view the trip distribution process as a well established approach for estimating O-D matrices, which has a long history and is hard to bring into question. In contrast, synthetic O-D generation is much less well known, and while many are not familiar with the details of the technique, most have heard of the difficulties arising from the under-specification problem. The above similarity indicates that planners should not question the validity of O-D matrices estimated synthetically any more than they question the validity of O-D matrices estimated from the trip distribution process. Second, if planners are presently utilizing O-D matrices derived from a trip distribution exercise, they can now recognize that providing additional information about en-route links within a synthetic O-D generation process can only enhance their matrix accuracy, especially if they use the trip distribution matrix as a seed.

It should be noted, however, that, trip distribution procedures can be utilized to estimate O-D tables for different trip purposes. Alternatively, the synthetic O-D approach cannot segregate trips by purpose unless the link counts are also classified by trip purpose. However, a possible approach to estimating purpose-specific O-D tables would be to utilize trip distribution procedures to derive O-D tables for the different trip purposes (where T_{ij}^k is the estimated number of trips between zones "i" and "j" for purpose "k" using trip distribution procedures). Scaling factors (c_{ij}^k) for each O-D pair can be computed using Equation 3. These O-D demand specific scaling factors can then be multiplied to the synthetic O-D table to generate purpose-specific O-D tables that are consistent with observed link flows.

$$C_{ij}^k = \frac{T_{ij}^k}{\sum_k T_{ij}^k} = \frac{T_{ij}^k}{T_{ij}} \quad [3]$$

MAXIMUM LIKELIHOOD FOR O-D MATRICES BASED ON TRIPS

Entropy maximization and information minimization techniques have been used to solve a number of transportation problems (Wilson, 1970). The application of the entropy maximization principles to the static O-D estimation problem was initially proposed by Willumsen (1978). Willumsen demonstrated that by maximizing the entropy, the most likely trip matrix could be estimated subject to a set of constraints.

Figure 1b illustrates a simple 5-link network in which there are 2 origin zones connected to 2 destination zones. Some of the directional links connecting them carry only two O-D pairs each (links 1, 2, 4 and 5), while link 3 carries all O-D pairs. If one considers the observed link flows of 40, 60, 100, 70, and 30 veh/h on links 1, 2, 3, 4, and 5, respectively, one can note that a number of different matrices can be applied to this network that replicate observed flows equally well. In general, if the number of trips from A to C is considered to be x , the number of trips between all other O-D pairs are automatically specified. The maximum likelihood approach, to selecting a synthetic O-D, therefore revolves around finding from all of the feasible O-D demands that are demonstrated in Figure 1 the most likely O-D matrix.

The trip-based approach to defining maximum likelihood considers that the overall trip matrix is made up of uniquely identifiable individual trip makers. In the case of Figure 1, it can be noted that there are a total of 100 trips in the matrix. If one now considers the 20 trip makers from A to C, one can show that there are $100!/(20!x80!)$ unique ways of drawing 20 trip makers from the population of 100 without considering the order of selection. This leaves 80 trip makers to draw from for the 10 trip makers from A to D, implying that there are $80!/(10!x70!)$ different ways of doing this. Similarly, there are $70!/(20!x50!)$ ways of selecting trip makers from B to C, and $50!/(50!x0!)$ to select trip makers from B to D. In total, and after some simplification, this creates for $100!/(20!x10!x20!x50!)$ ways of creating a simple 20, 10, 20 and 50 trip matrix. In general, this number is computed using Equation 4a. The first plot in Figure 2 illustrates, for all of the matrices identified in Table 3, their associated likelihood. It is clear from this figure that the most likely matrix is one where $x=28$.

$$\text{Max. } Z(T_{ij}) = \frac{T!}{\prod_{ij} (T_{ij}!)} \quad [4a]$$

The above formulation does not take into account any prior information, from for example a previous survey. A seed matrix can be constructed from a previous survey. While the seed matrix does not necessarily have to satisfy the observed link flows, the seed matrix can be utilized to expand the maximum likelihood function as shown in Equation 4b. It can be noted that the likelihood of an individual trip from i to j is $t_{ij}/\sum t_{ij}$, based on the above seed matrix. Consequently, the probability of T_{ij} trips being drawn is $(t_{ij}/\sum t_{ij})^{T_{ij}}$. It can be shown, in Figure 2, that the use of the above non-uniform seed matrix yields a different maximum likelihood O-D matrix with a different objective function. Conversely, when a uniform seed matrix is applied to the 5-link network of Figure 1, it can be shown to have no impact on the final optimum O-D solution; however the value of the objective function is different. This finding is unique to only certain network types, and does not hold in general. For reasons that will be discussed later, it will be considered that the formulation including the seed is the correct one, when the formulation with and without the seed differ.

$$\text{Max. } Z(T_{ij}, t_{ij}) = \frac{T!}{\prod_j (T_{ij}!)} \prod_{ij} \left(\frac{t_{ij}}{\sum_j t_{ij}} \right)^{T_{ij}} \quad [4b]$$

MAXIMUM LIKELIHOOD BASED ON VOLUME COUNTS

An alternative maximum likelihood formulation considers that the basic unit of observation is a volume count. In the network of Figure 1, for example, it can be noted that there are 300 volume counts (30+70+100+40+60). If one now considers a potential solution matrix, one can show that the number of ways in which these volume counts can be drawn from the original set of 300 is as shown in Equation 5.

$$\text{Max. } Z(T_{ij}, t_{ij}) = \prod_a \frac{V_a!}{\prod_{ij} (T_{ij} p_{ij}^a)} \prod_{ij} \left(\frac{t_{ij} p_{ij}^a}{V_a} \right)^{T_{ij} p_{ij}^a} \quad [5]$$

Despite the somewhat different structure of Equation 4, it can be shown that for the network in Figure 1, this objective function identifies the same O-D matrix as being the most likely as either one of the two earlier trip based formulations. However, this is not a general finding. Specifically, in some cases where the trip based formulation without an explicit reference to a seed differs from the one which includes the seed, albeit a uniform one, the volume based formulation is only consistent with the latter solution (Equation 4b). For this reason, the trip based formulation, including the explicit accounting of the seed matrix, is viewed to be the bench mark against which all other solutions and formulations should be compared.

PRESENCE OF FLOW CONTINUITY

The above formulations of objective functions for expressing likelihood require additional constraints in order to be complete (Van Zuylen and Willumsen, 1978 and Willumsen 1980). The simplest of these constraints indicate that the sum of all trips crossing a given link must be equal to the link flow on that link, as indicated in Equation 6. As will be shown later, the simplest mechanism, for including the above constraints in the earlier objective functions, is to utilize Lagrange multipliers. These multipliers permit an objective function with equality constraints to be transformed into an equivalent unconstrained objective function.

$$V_a = \sum_{ij} T_{ij} p_{ij}^a \quad \forall a \quad [6]$$

This simple set of equality constraints, while making the formulation complete, may at times also render the problem infeasible. For example, consider the network of Figure 1 that the observed flows are 30, 70, 110, 40 and 60. In this case, there is no set of O-D matrices that can match the observed link flows, as there is no continuity of flow at the nodes interior to the network for these flows.

A more general formulation that is proposed in this paper, therefore, is to request that the error in the link flow constraints be minimized, rather than eliminated. In other words, rather than finding the most likely O-D that exactly replicates the observed link flows, the problem is re-formulated as finding the most likely O-D matrix from among all of those that come equally close to matching the link flows. One expression that is proposed to capture the error to be minimized is shown in Equation 7a, and is subject to the flow continuity constraints in Equation 7b. The constraints in Equation 7b can be introduced in Equation 7a to yield an unconstrained objective function, yielding a set of complementary link flows V'_a . These complementary flows are those which deviate the least from the observed link flows, while satisfying link flow continuity. Given that these complementary link flows do satisfy link continuity, they can now be added in as rigid equality constraints to the objective function 4a or 4b, and be guaranteed to yield a feasible solution.

$$\text{Min. } Z(T_{ij}) = \sum_a (V_a - V'_a)^2 \quad [7a]$$

$$V'_a = \sum_{ij} T_{ij} p_{ij}^a \quad \forall a \quad [7b]$$

$$\text{Min. } Z(T_{ij}) = \sum_a \left(V_a - \sum_{ij} T_{ij} p_{ij}^a \right)^2 \quad [7c]$$

Alternatively, one can incorporate Equations 7b into Equation 7a. This new expression, which is shown as Equation 7c, is one which should be minimized concurrently to maximizing objective function 4a or 4b. Unfortunately, it is not easy to combine one expression that desires to maximize likelihood with another that desires to minimize link flow error, as Lagrangian can only add equality constraints to a constrained objective function. A proposed solution to this problem involves taking the partial derivatives of Equation 6c with respect to each of the trip cells that are to be estimated, as demonstrated in Equation 8a. This yields as many equations as there are trip cells, as shown in Equation 8b. Furthermore, setting these derivatives equal to 0 is equivalent to minimizing Equation 8a. However, while equation 7a could not be added to the maximum likelihood objective function, the equalities in Equation 8b can. This produces an unconstrained objective function that always yields a feasible solution, as is shown in Equation 9.

The net result, of the above process, is to suggest that most synthetic O-D generation problems consist of two sub-problems. One of these involves finding a new set of complementary link flows that do produce link flow continuity, at which point the maximum likelihood problem can be solved as before. Alternatively, one can compute the partial derivatives, that will yield link flow continuity, while deviating by the least amount from the observed link flows, and then utilize them directly in the maximum likelihood formulation using Lagrangian multipliers. Both solutions can be shown to yield identical results.

$$\frac{\partial}{\partial T_{ij}} Z(T_{ij}) = \frac{\partial}{\partial T_{ij}} \sum_a \left(V_a - \sum_{ij} T_{ij} p_{ij}^a \right)^2 = 0 \quad \forall i, j \quad [8a]$$

$$0 = 2 \left(\sum_a (V_a \cdot p_{ij}^a) - \left(\sum_a p_{ij}^a \left(\sum_{xy} T_{xy} p_{xy}^a \right) \right) \right) \quad \forall i, j \quad [8b]$$

$$\text{Max. } \frac{T!}{\prod_{ij} T_{ij}} \prod_{ij} \left(\frac{t_{ij}}{t} \right)^{T_{ij}} - \sum_{ij} \left(\lambda_{ij} \cdot 2 \left(\sum_a (V_a \cdot p_{ij}^a) - \left(\sum_a p_{ij}^a \left(\sum_{xy} T_{xy} p_{xy}^a \right) \right) \right) \right) \quad [9]$$

$$\text{Where: } T = \sum_{ij} T_{ij} \text{ and } t = \sum_{ij} t_{ij}$$

NUMERICAL SOLUTION APPROXIMATIONS

Stirling's Approximation

A first challenge with maximizing Equation 8 is that it yields very large numbers that are difficult to work with. Further more, as it is common to maximize objective functions by taking their derivatives, and as it is more difficult to contemplate the derivative of a discontinuous expression, such as those including factorials, a simple approximation is made. This approximation involves taking the natural logarithm of either objective function Equation 4a or 4b. Taking the natural logarithm of the objective function both makes the output easier to handle, but allows us to use Stirling's approximation as a convenient continuous equivalent to the term $\ln(x!)$, as demonstrated in Equation 10.

The resulting converted objective function using the Stirling approximation on the original objective function of Equation 4b is presented as Equation 11. Expanding the various terms, Equation 12 is derived. When Equation 12 is augmented with the previously mentioned partial derivatives that minimize link flow error, Equation 13 emerges. This equation, when solved, yields the most likely O-D matrix of all of those matrices that come equally close to matching the observed link flows.

The above set of non-linear equations can be solved numerically in a number of ways using a variety of standard numerical analysis software packages. However, as for large networks the number of equations and unknowns becomes extremely difficult. Consequently, a special purpose equation solver has been developed, called QUEENSOD. This solver fully optimizes the objective function of Equation 13. The formulation of Equation 13 makes a single approximation namely the Stirling approximation, which has been shown to produce errors less than 1% for the range of values and derivatives being typically considered in the problem. A sample application of the QUEENSOD software is presented later in the paper.

It should be noted that the objective function of Equation 13 is composed of two components. The first being the error between the field observed flows and the flows that satisfy flow continuity with minimum difference from observed flows. The second component represents the likelihood of an O-D matrix table. The objective is to find the O-D matrix with the maximum likelihood. In the case that the seed matrix is the optimum matrix the likelihood component resolves to zero.

$$\ln(T!) = T \ln T - T \quad [10]$$

$$\text{Max. } T \ln\left(\frac{T}{t}\right) - T - \sum_{ij} \left(T_{ij} \ln\left(\frac{T_{ij}}{t_{ij}}\right) - T_{ij} \right) - \sum_{ij} \left(\lambda_{ij} \cdot 2 \left(\sum_a (V_a \cdot p_{ij}^a) - \left(\sum_a p_{ij}^a \left(\sum_{xy} T_{xy} p_{xy}^a \right) \right) \right) \right) \quad [11]$$

$$T \ln\left(\frac{T}{t}\right) - T - \sum_{ij} \left(T_{ij} \ln\left(\frac{T_{ij}}{t_{ij}}\right) - T_{ij} \right) = T \ln\left(\frac{T}{t}\right) - \sum_{ij} T_{ij} \ln\left(\frac{T_{ij}}{t_{ij}}\right) \quad [12]$$

$$\text{Max. } T \ln\left(\frac{T}{t}\right) - \sum_{ij} T_{ij} \ln\left(\frac{T_{ij}}{t_{ij}}\right) - \sum_{ij} \left(\lambda_{ij} \cdot 2 \left(\sum_a (V_a \cdot p_{ij}^a) - \left(\sum_a p_{ij}^a \left(\sum_{xy} T_{xy} p_{xy}^a \right) \right) \right) \right) \quad [13]$$

Other Approximations

While QUEENSOD solves the full objective function, utilizing only Stirling's approximation, others have made more drastic approximations. Some of these, such as those proposed by Willumsen (1980, 1981, and 1984), are widely utilized but can be shown to introduce considerable error, as will be indicated next. It should be noted that a full description of the implication of each approximation on the solution is beyond the scope of this paper and will be presented in a forthcoming publication.

The most common approximation that is applied, after Stirling's approximation is invoked, is to consider that the total number of trips in the network T is constant, and to drop this term yielding Equation 14. Unfortunately, as is shown in

Figure 2 for the 2-link network illustrated in Figure 1, dropping T for networks in which the total number of trips is not constant, can yield a very different optimum solution.

$$\text{Max. } Z(T_{ij}, t_{ij}) = -\sum_{ij} \left(T_{ij} \text{Ln} \left(\frac{T_{ij}}{t_{ij}} \right) - T_{ij} \right) \quad [14]$$

The next most common approximation is to simplify Equation 14 to become Equation 15a by adding a constant term $\sum t_{ij}$. A further approximation can then be applied using equation 15b to yield Equation 15c, and eventually 15d. A simple substitution of t_{ij} to T_{ij} finally yields Equation 15e, which can then be solved using simple weighted linear regression. Unfortunately, while the solution is much easier to obtain, it has two main faults. The first of these is that the solutions found using regression deviate considerably from those found using either the original objective function or its Stirling's approximation. Secondly, the approximation is such that at times it can yield negative values for the number of trips between a given O-D pair, which is meaningless.

$$\text{Max. } Z(T_{ij}, t_{ij}) = -\sum_{ij} \left(T_{ij} \text{Ln} \left(\frac{T_{ij}}{t_{ij}} \right) - T_{ij} + t_{ij} \right) \quad [15a]$$

$$\text{Ln}(x) = \left(\frac{x-1}{x} \right) + \frac{1}{2} \left(\frac{x-1}{x} \right)^2 + \frac{1}{3} \left(\frac{x-1}{x} \right)^3 + \dots \quad [15b]$$

$$\text{Max. } Z(T_{ij}, t_{ij}) = -\sum_{ij} \left(T_{ij} \left(\frac{\frac{T_{ij}}{t_{ij}} - 1}{\frac{T_{ij}}{t_{ij}}} \right) + \frac{T_{ij}}{2} \left(\frac{\frac{T_{ij}}{t_{ij}} - 1}{\frac{T_{ij}}{t_{ij}}} \right)^2 - T_{ij} + t_{ij} \right) \quad [15c]$$

$$\text{Min. } Z(T_{ij}, t_{ij}) = \sum_{ij} \left(\frac{1}{2T_{ij}} (T_{ij} - t_{ij})^2 \right) \quad [15d]$$

$$\text{Min. } Z(T_{ij}, t_{ij}) = \sum_{ij} \left(\frac{1}{2t_{ij}} (T_{ij} - t_{ij})^2 \right) \quad [15e]$$

SAMPLE MODEL APPLICATION

The QUEENSOD software estimates the maximum likelihood O-D matrix by solving Equation 13 numerically. The numerical solution begins by building a minimum path tree and performing an all-or-nothing traffic assignment of the seed matrix. A relative or absolute link flow error is computed depending on user input. Using the link flow errors O-D adjustment factors are computed and utilized to modify the seed O-D matrix. The adjustment of the O-D matrix continues until one of two criteria are met, namely the change in O-D error reaches a user-specified minimum or the number of iterations criterion is met. If additional trees are to be considered, the model builds a new set of minimum path trees and shifts traffic gradually to the second minimum path tree. The minimum objective function for two trees is computed in a similar fashion as described for the single tree scenario. The process of building trees and finding the optimum solution continues until all possible trees have been explored.

The proposed numerical solution ensures that in the case that the seed matrix is optimum no changes are made to the matrix, as was the case in the formulation of Equation 13. In addition, the use of the seed matrix as a starting point for the search algorithm ensures that the optimum solution resembles the seed matrix as closely as possible while minimizing the link flow error. In other words, the seed matrix biases the solution towards the seed matrix.

In order to demonstrate the applicability of the QUEENSOD software, a sample application to a 3500-link network of the Bellevue area in Seattle is presented. Other applications of the QUEENSOD software are described in detail in the literature (Rakha *et al.*, 1998 and 2000). The O-D demand for the Bellevue network was calibrated to AM peak Single

Occupancy Vehicle (SOV) and High Occupancy Vehicle (HOV) flows. The seed matrix was created using the standard four-step transportation planning process by applying the EMME/2 model. The Seattle network was converted from EMME/2 format to INTEGRATION format.

The calibration of the O-D demand to tube and turning movement counts was conducted using the QUEENSOD software using the planning trip distribution O-D matrix as the seed solution. The calibration resulted in a high level of consistency between estimated and field observed link flow counts (coefficient of determination of 0.98 between the estimated and observed flows), as illustrated in Figure 3. Figure 3 demonstrates that in calibrating the O-D table to observed traffic counts, the trip distribution O-D matrix (seed matrix) was modified significantly (coefficient of determination of 0.56 between trip distribution and synthetic O-D table). Consequently, it is evident that a modification of the trip distribution matrix was required in order to better match observed link and turning movement counts. It should be noted however, that the total number of trips was increased by only 4 percent as a result of the synthetic O-D calibration effort. Consequently, the illustrated calibration effort resulted in a significant modification of the trip table with minor modification to the total number of trips.

CONCLUSIONS

The paper provided a comprehensive overview of the entire class of formulations and most recognized solutions for estimating Origin-Destination (O-D) demand tables. Specifically, the paper compared trip distribution formulations to less known synthetic O-D estimation solution techniques. The paper demonstrated that the trip distribution gravity model is a subset of the maximum likelihood solution to the synthetic O-D problem. Finally, the paper proposed a numerical solution to the maximum likelihood synthetic O-D problem that only requires the Stirling's approximation and does not require flow continuity in observed flows. The proposed solution, which has been implemented in the QUEENSOD software, has been applied to networks with more than 1,000 zones and 5,000 links on a PC, where it typically requires no more than 1 hour to solve problems of this size. The solutions obtained by QUEENSOD reflect multi-path routings, consider correctly that the total number of trips in the network may not be constant, and properly reflect the role of the seed matrix. The model can also be applied to deal with problems where the routes are not known a priori.

ACKNOWLEDGEMENTS

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NOTATION

T_{ij}	= Estimated number of trips between zones i and j for the analysis period for all trip purposes
T_{ij}^k	= Estimated number of trips between zones i and j for the analysis period for trip purpose " k "
c_{ij}^k	= Proportion of trips between zones " i " and " j " for purpose " k " = T_{ij}^k/T_{ij}
P_i	= Total number of trips produced by zone i for the analysis period
A_j	= Total number of trips attracted to zone j for the analysis period
F_{ij}	= Impedance function for travel between zones i and j during the analysis period (typically inverse function of travel time)
K_{ij}	= Socioeconomic adjustment factor between zones i and j
t_{ij}	= Seed trips between zones i and j
T	= Total number of trips ($\sum T_{ij} = \sum P_i = \sum A_j$)
V_a	= Actual observed link volume on link " a "
V'_a	= Volume on link " a " that are closest to V_a that satisfy flow continuity

REFERENCES

Rakha H., Van Aerde M., Bloomberg L., and Huang X. (1998), *Construction and Calibration of a Large-Scale Micro-Simulation Model of the Salt Lake Area*, Transportation Research Record, No. 1644, pp. 93-102.

- Rakha H., A. Medina, H. Sin, F. Dion, and J. Jenq (2000). "Coordination of Traffic Signals across Jurisdictional Boundaries: Field and Modeling Results." Presented at the Transportation Research Board 79th Annual Meeting, Washington DC, January, [Paper # 00-1560].
- M. Van Aerde & Assoc., Ltd. (2002). QUEENSOD® Rel. 2.10 User's Guide: Estimating Origin Destination Traffic Demands from Link Flows. M. Van Aerde & Assoc., Ltd.
- Van Zuylen J. H and Willumsen L. G. (1980). The most likely trip matrix estimated from traffic counts. *Transpn. Res.* 14B, 281-293.
- Willumsen L. G. (1978) *Estimation of an O-D matrix from traffic counts: A review*. Institute for Transport Studies, Working paper no. 99, Leeds University.
- Willumsen L. G. (1981) Simplified transport models based on traffic counts. *Transportation* 10, 257-278
- Willumsen L. G. (1984) Estimating time-dependent trip matrices from traffic counts. *Ninth International Symposium on Transportation and Traffic Theory*, pp. 397-411. Utrecht: VNU Science Press.
- Wilson A. G. (1970) *Entropy in Urban and Regional Modelling*. Pion, London.

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Table 1: Complementary Nature of Synthetic O-D Process and Trip Distribution (18-Link Network)

Link	Link Type	Link Count	Information in Trip Distribution Process	Information in Synthetic O-D Process
1	Zone Connector	100	100	-
2	Zone Connector	200	200	-
3	Zone Connector	200	200	-
4	Zone Connector	200	200	-
5	Zone Connector	300	300	-
6	Zone Connector	300	300	-
7	Zone Connector	400	400	-
8	Zone Connector	300	300	-
11	En-route Link	35	-	35
12	En-route Link	61	-	61
13	En-route Link	139	-	139
14	En-route Link	65	-	65
15	En-route Link	269	-	269
16	En-route Link	296	-	296
17	En-route Link	80	-	80
18	En-route Link	165	-	165

Table 2: Comparison of Trip Distribution and Synthetic O-D Solutions (18-Link Network)

Trip Distribution Solution				
	A	B	C	D
A	0.00	34.95	11.71	53.34
B	60.63	0.00	79.77	59.60
C	22.93	90.04	0.00	187.04
D	116.45	75.00	208.54	0.00
Synthetic O-D Solution				
	A	B	C	D
A	0.00	34.75	13.76	57.15
B	59.78	0.00	82.54	54.89
C	25.17	92.71	0.00	184.89
D	118.74	70.75	207.26	0.00

Table 3: Feasible Solutions (5-Link Network)

Demand Scenario	O-D Demand (T_{ij})			
	T_{AC}	T_{AD}	T_{BC}	T_{BD}
0	10	30	60	0
1	11	29	59	1
2	12	28	58	2
3	13	27	57	3
4	14	26	56	4
5	15	25	55	5
6	16	24	54	6
7	17	23	53	7
8	18	22	52	8
9	19	21	51	9
10	20	20	50	10
11	21	19	49	11
12	22	18	48	12
13	23	17	47	13
14	24	16	46	14
15	25	15	45	15
16	26	14	44	16
17	27	13	43	17
18	28	12	42	18
19	29	11	41	19
20	30	10	40	20
21	31	9	39	21
22	32	8	38	22
23	33	7	37	23
24	34	6	36	24
25	35	5	35	25
26	36	4	34	26
27	37	3	33	27
28	38	2	32	28
29	39	1	31	29
30	40	0	30	30

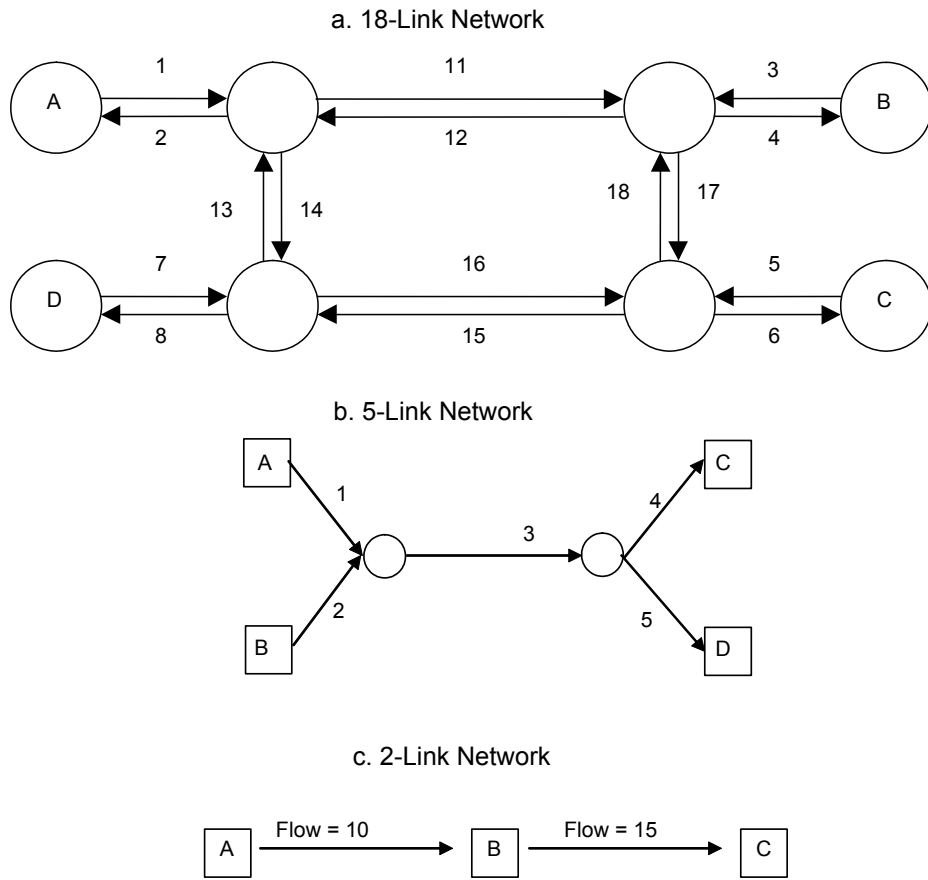
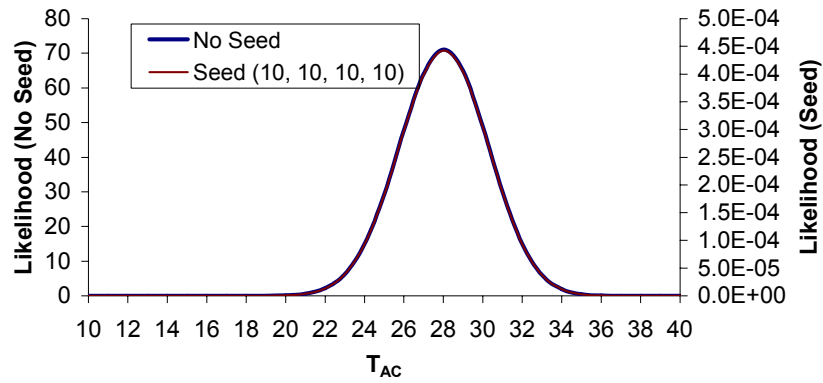
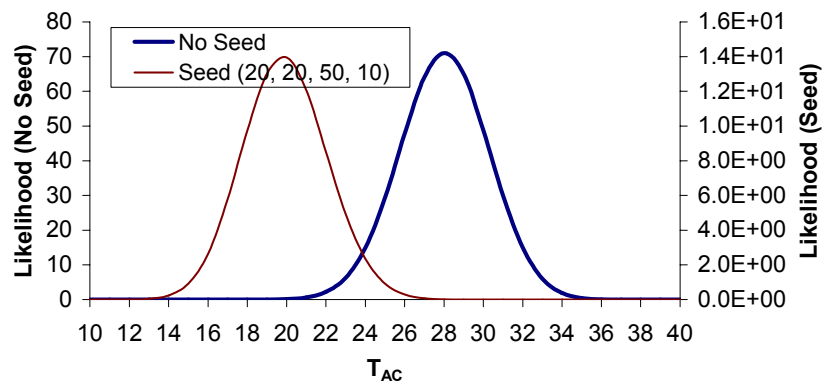


Figure 1: Sample Networks Configurations

a. 5-Link Network - Uniform Seed



b. 5-Link Network - Non-Uniform Seed



c. 2-Link Network

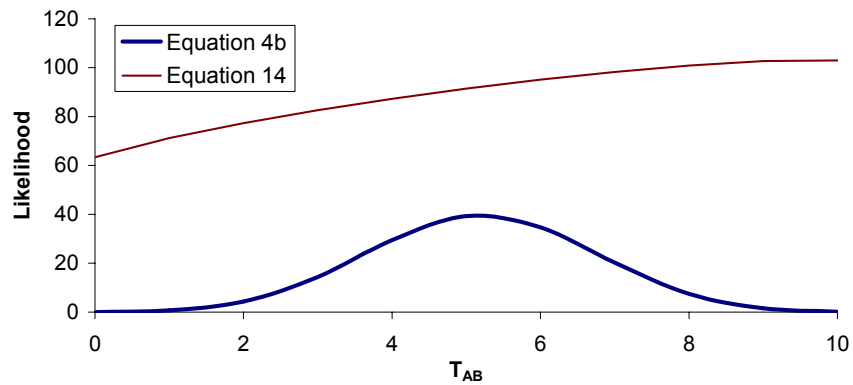


Figure 2: Variation in Likelihood as a Function for Different Scenarios

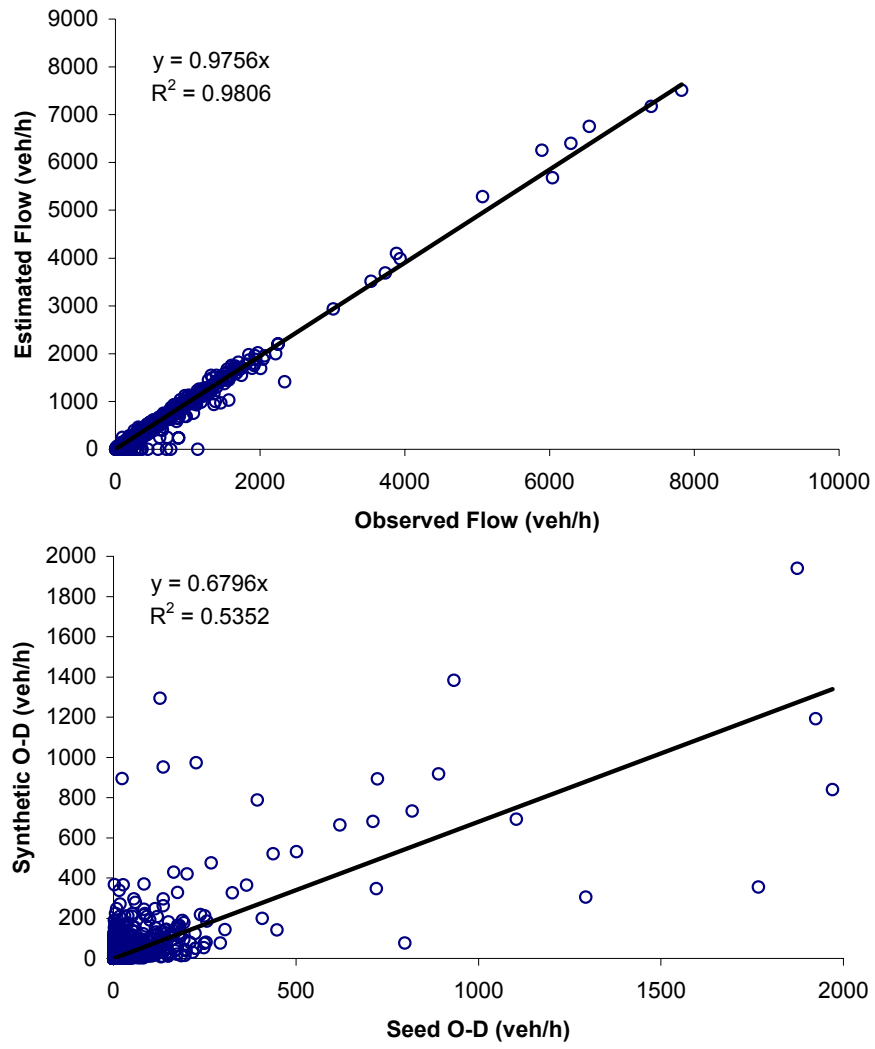


Figure 3: Example Application of QUEENSOD to the Bellevue Network in Seattle