WORKING GROUP FOR THE COMPLEXITY OF LEARNING TO REASON PROBABILISTICALLY

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There are several critical aims that guide our work together. In particular, members are interested in examining: (1) mathematical and psychological underpinnings that foster or hinder students' development of probabilistic reasoning, (2) the influence of experiments and simulations in the building of ideas by learners, particularly with emerging technology tools, (3) learners' interactions with and reasoning about data-based tasks, representations, models, socially situated arguments and generalizations, (4) the development of reasoning across grades, with learners of different cultures, ages, and social backgrounds, and (5) the interplay of statistical and probabilistic reasoning and the complex role of key concepts such as variability and data distributions. Recent foci in the working group have been to understand: (1) students' and teachers' reasoning when simulating probability experiments with hands-on materials and computer tools, and (2) connections between probability and statistical concepts such as inference and variability. At PME-NA 27 in Roanoke, Virginia, the group will build on discussions and the elaboration of the research agenda that it began at PME-NA 26 in Toronto. To stimulate and extend discussions, members of the group will be invited to show video clips in order to engage the group in data analysis that can inform work on (1) designing tasks that elicit students' probabilistic thinking, (2) understanding how students learn to reason probabilistically, and (3) outlining implications for teaching, learning, and research. Through these discussions, group members will refine previously posed research questions as well as elaborate additional questions. The group will also make preliminary designs for cross-national, collaborative research to be conducted in 2006. The working-group organizers plan to solicit several papers emerging from the cross-national collaborative work of group members that will lead to a set of papers that describe our work.

Nature and Topic of the Working Session

This Working Group was formed at PME-NA 20 (Maher, Speiser, Friel, & Konold, 1998) and has convened annually at PME-NA each of the past six years (see Maher & Speiser, 1999; 2001; 2002; Speiser, 2000; Stohl & Tarr, 2003; Tarr & Stohl, 2004). During the joint meeting of PME-NA 25 and PME 27 in 2003 (Hawaii, USA) and PME-NA 26 in Toronto, Canada, we expanded our working group to include many more international researchers across 12 different countries. Through shared research, rich and engaging conversations, and analysis of instructional tasks, we continually seek to understand how students learn to reason probabilistically. There are several critical aims that guide our work together. In particular, members are interested in examining: (1) mathematical and psychological underpinnings that foster or hinder students' development of probabilistic reasoning, (2) the influence of experiments and simulations in the building of ideas by learners, particularly with emerging technology tools, (3) learners' interactions with and reasoning about data-based tasks, representations, models, socially situated arguments and generalizations, (4) the development of

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reasoning across grades, with learners of different cultures, ages, and social backgrounds, and (5) the interplay of statistical and probabilistic reasoning and the complex role of key concepts such as variability and data distributions. Recent foci in the working group have been to understand: (1) students' and teachers' reasoning when simulating probability experiments with hands-on materials and computer tools, and (2) connections between probability and statistical concepts such as inference and variability.

Background on Probabilistic Reasoning

The ways in which students reason about the likelihood of an event can be considered in terms of an objective or subjective view of probability (e.g., see Batanero, Henry, & Parzysz, 2005; Borovcnik, Bentz, & Kapadia, 1991). One cannot precisely determine whether a 4 will appear when rolling a regular six-sided die because of the complex physics involved (e.g., the speed and angle at which the die is thrown, the initial spin of the die, air resistance – see Wolfram, 2002). In the presence of this uncertainty, the construct of probability is formed as a theoretical model of the event. In an *objectivist* perspective, probability is viewed as an inherent property of the event and can be well estimated either through a classical or frequentist approach. We can use a classical Laplacean approach to embody the complexities of the physics and apparent (and probably imperfect) symmetry of the die *a priori* tossing the die and express the probability of rolling a 4 as 1/6. This probability of 1/6 is an estimate of the actual theoretical probability that is unknown to us. If one rolls a die a given number of times, a frequentist approach can be used to hypothesize the probability in terms of the theoretical limit of the observed proportion of 4's as the number of trials tends to infinity. But again, this estimate is bounded by real world constraints and can only describe the probability of getting a 4 based on a finite set of die rolls. A repeated finite set of die rolls would most likely yield a different experimental estimate of the actual probability and may in fact allow one to change the estimate of the probability based on new data.

In a *subjectivist* perspective, probability is viewed as a condition of the information known to the individual assigning the probability and not an objective property of the given event. Thus, two people may assign different probabilities to the same event based on different *a priori* information, even after they observe the same empirical data *a posteriori* trials being conducted. For example, one student might recall instances where there were few outcomes of 4 and thus might infer that all outcomes are not equally likely. Another student might believe that all outcomes on a die are equally likely based on their previous experience of not being able to predict the outcome of rolling a die and noticing no distinct pattern in any number being "harder to get." Upon conducting repeated trials and observing that there were relatively fewer 1's and 6's and more 4's, the first student might state that all outcomes are equally likely since this set of data was different than his belief about 4's from his prior experience. The second student may be perturbed by the low number of 6's as compared to her belief that none of the numbers should be "harder to get" and subsequently believe the die is biased.

The *law of large numbers* is used to interpret empirical results in relation to theoretical probabilities and, thus supports the viability that an estimated probability from a frequentist approach will be reasonably close to the theoretical probability. This principle states that the probability of a large difference between the relative frequency of an outcome and the theoretical probability limits to zero as more trials are collected. Even after a large number of trials, it is possible to have a relative frequency substantially different than the theoretical probability.

A frequentist approach to probability, grounded in the law of large numbers, has only recently made its way into curricular aims in schools (Jones, 2005). Teachers are encouraged to use an empirical introduction to probability by allowing students to experience repeated trials of the same event, either with concrete materials or through computer simulations (e.g., Batanero, Henry, & Parzysz, 2005; National Council of Teachers of Mathematics [NCTM], 2000; Parzysz, 2003). In these types of curricula, a theoretical model of probability based on a classical approach is not the starting point. Rather, a theoretical model is constructed based on observing that the relative frequencies of an event from a repeated random experiment stabilize as the number of trials or sets of trials (different samples) increases. However, there is general agreement that research on students' probabilistic reasoning has been lacking sufficient study of students' understanding of the connection between observations from empirical data (probability in reality) and a theoretical model of probability (e.g., Jones, 2005; Parzysz, 2003).

Guiding Framework for Our Discussions

As a way of framing the discussions for the working group, we are utilizing a conceptual framework developed by Lee, Rider, and Tarr (2005) which was extended from earlier work by Stohl and Tarr (2002). This framework can provide the members of the working group a common starting place and a way of talking about students' probabilistic reasoning. In no way is the group restricted to using this framework, and in fact we hope modifications, extensions, and improvements will emerge out of group discussions and subsequent research efforts.

Considering the bi-directional model shown in Figure 1, students may start from the theoretical side of the model and begin to reason using an image of the theoretical probability of each event developed from either an objective classical view or a more subjective view based on their experiences and knowledge. Their initial assumptions of the probability provide an image of what students expect to observe in empirical data. They may then compare the results (e.g., frequencies or relative frequencies of an event) against their mental image and initial hypothesis about the probability of that event (Watson & Kelly, 2004). Noticing patterns in the data may make them call into question the prior assumptions, or they may not believe the data varies enough from their mental image to contradict their initial assumption. Their reasoning may then lead them to decide to collect and analyze more data to again test the reasonableness of the match between their mental image of the hypothesized theoretical probability and results from repeated empirical trials.

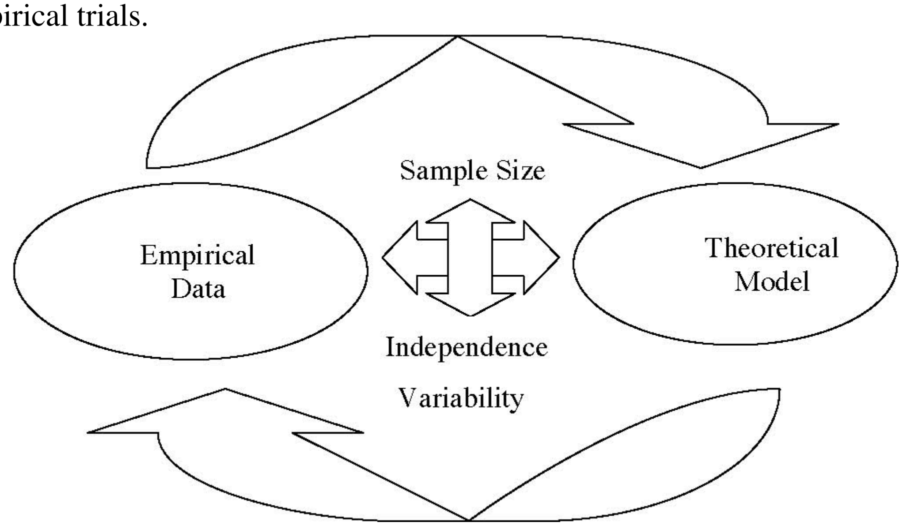

Figure 1. Bi-directional model

Starting from the empirical side of the model, students may reason about the probability of an event where they have no prior experience with the phenomena or can not use a classical approach (e.g., how likely is it that a tack will land on its side when dropped on the floor). Thus, they may start by examining empirical data and using the relative frequencies from that data to inform a hypothesis regarding the underlying theoretical probabilities. The first hypothesis about theoretical probability allows students to form a mental image of the expected results in future data. Students may then use their mental image to inform how (or whether) to collect empirical data about the phenomena.

The robustness of students' reasoning from empirical data back to their initial assumption of the theoretical probability is influenced by the sample size, understanding the independence of trials, and variability of their data. Students need to consider that different trials and different sets of trials (samples) are independent of one another and variability among individual trials and samples is to be expected. They also need to coordinate conceptions of independence and variability with the role of sample size in the design of data collection and interpretation of results. Relative frequencies from larger samples are likely to be more representative of the theoretical probabilities while smaller samples may offer more variability and be less representative. For example, ten rolls may yield no 3's and such data may support a child's notion that rolling a 3 is an improbable (or even impossible) event, although such a claim would likely not be made by someone who had made a more robust connection between relative frequency in empirical data and theoretical probability, and the importance of sample size.

Summary of Activities from 2004

Seventeen researchers (faculty and graduate students) from the United States, Canada, Mexico, and Israel met During PME-NA 26 in Toronto. After analyzing a video of students' work on a computer-based simulation task (Schoolopoly task, see Stohl & Tarr, 2002; Lee, Rider, & Tarr, 2005), the group discussed the different aspects of students' thinking when they are trying to generate and analyze empirical data to make inferences about an unknown probability distribution. This discussion led to different participants expressing interest in conducting various pilot research studies during Spring 2005. Some of the ideas for follow-up research included:

- What are the longitudinal benefits and effects of students' engagement in using simulation techniques to approach probability and statistics tasks— particularly following students from middle school through Advanced Placement (AP) Statistics?
- How do students who are at the end of an Advanced Placement (AP) Statistics course and have been traditionally taught with an emphasis on theoretical statistics and probability apply their understandings to a task like Schoolopoly where students must generate and analyze data to make inferences about an unknown probability distribution?
- How do students' reasoning about the design and results of probability simulations differ when they use hands-on tools (e.g., coins, dice, spinners, bags of marbles) and computer-based tools?
- What role does students' agency play in their understanding of probability concepts when given an open-ended tool like Probability Explorer to design experiments, generate data, and analyze results to make inferences about unknown distributions?

Planned Activities for 2005 Meeting

At PME-NA 27 in Roanoke, Virginia, the group will build on discussions and the elaboration of the research agenda that it began at PME-NA 26 in Toronto. To stimulate and extend discussions, members of the group will be invited to show video clips in order to engage the group in data analysis that can inform work on (1) designing tasks that elicit students' probabilistic thinking, (2) understanding how students learn to reason probabilistically, and (3) outlining implications for teaching, learning, and research. Through these discussions, group members will refine previously posed research questions as well as elaborate additional questions. The group will also make preliminary designs for cross-national, collaborative research to be conducted in 2006. The working-group organizers plan to solicit several papers emerging from the cross-national collaborative work of group members that will lead to a set of papers that describe our work. These papers could be part of a monograph, journal special issue, and many joint presentations at future conferences.

References

Batanero, C., Henry, M., & Parzysz, B. (2005).The nature of chance and probability. In G. A. Jones (Ed.) *Exploring probability in school: Challenges for teaching and learning* (pp. 15-37). Netherlands: Kluwer Academic Publishers.

Borovcnik, M., Bentz, H.-J., & Kapadia, R. (1991). A probabilistic perspective. In R. Kapadia & M. Borovcnik (Eds.), *Chance encounters: Probability in education* (pp. 27-71). Boston: Kluwer Academic Publishers.

Jones, G. A. (2005). Reflections. In G. A. Jones (Ed.) *Exploring probability in school: Challenges for teaching and learning,* (pp. 375-379). Netherlands: Kluwer Academic Publishers.

Lee, H. S., Rider, R. L., & Tarr, J. E. (2005). Making connections between empirical data and theoretical probability: Students' generation and analysis of data in a technological environment. Manuscript under review for publication.

Maher, C. A., & Speiser, R. (1999). The complexity of learning to reason probabilistically. In F. Hitt, M. Santos (Eds.), *Proceedings of the Twenty-First Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 181-186). Cuernavaca, Morelos, Mexico: Center for Research and Advanced Studies.

Maher, C. A., & Speiser, R. (2001). The complexity of learning to reason probabilistically. In R. Speiser, C. A. Maher, & C. N. Walter (Eds.), *Proceedings of the Twenty-Third Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 73-78). Columbus, OH: ERIC Clearinghouse for Mathematics, Science, and Environmental Education.

Maher, C. A., & Speiser, R. (2002). The complexity of learning to reason probabilistically. In D. S. Mewborn, P. Sztajn, D. Y. White, H. G. Wiegel, R. L. Bryant, & K. Nooney (Eds.), *Proceedings of the Twenty-Fourth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 103-108). Columbus, OH: ERIC Clearinghouse for Mathematics, Science, and Environmental Education.

Maher, C. A., Speiser, R., Friel, S. & Konold, C. (1998). Learning to reason probabilistically. In S. Berenson, K. Dawkins, M. Blanton, W. Columbe, J. Kolb, & K. Norwood (Eds.), *Proceedings of the Twentieth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, v. 1, (pp. 82-87). Columbus, OH: ERIC Clearinghouse for Mathematics, Science, and Environmental Education.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Parzysz, B. (2003). *From frequency to probability. Some questions posed by the new French senior high school curricula*. Paper presented at the 54th International Statistical Institute, Berlin, Germany. Retrieved March 1, 2005 from: http://www.stat.auckland.ac.nz/~iase/publications.php?show=3 .

Speiser, R. (2000). The complexity of learning to reason probabilistically. In M.L. Fernandez (Ed.), *Proceedings of the Twenty-Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 45-50). Columbus, OH: ERIC Clearinghouse for Mathematics, Science, and Environmental Education.

Stohl, H. (1999-2002). *Probability Explorer* [software application]. Published by Author. Available at http://www.probexplorer.com .

Stohl, H. & Tarr, J. E. (2002). Developing notions of inference with probability simulation tools. *Journal of Mathematical Behavior 21(3),* 319-337.

Stohl, H. & Tarr, J.E. (2003). The complexity of learning to reason probabilistically. In N. A. Pateman, B. J. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 2003 Joint Meeting of PME and PMENA* (p. 178). Honolulu, HI: Center for Research and Development Group.

Watson, J. M., & Kelly, B. A. (2004). Statistical variation in a chance setting: A two year study. *Educational Studies of Mathematics 57*(1), 121-144. Wolfram, S. (2002). *A new kind of science*. Champaign, IL: Wolfram media, Inc.